# Caratheodory solutions of fuzzy differential equation

Presented by:
A.ELBADRAOUI
**Génie Mathématiques et Applications**

Université Sultan Moulay Slimane
Faculté des Sciences et Thechniques
Département de Mathématiques
Béni Mellal

16 Janvier 2017

Fuzzy sets are considered with respect to a nonempty base set $X$ of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, such that

$$1 \begin{cases} u(x) = 0 & \text{corresponding to non-membership} \\ 0 < u(x) < 1 & \text{to partial membership} \\ u(x) = 1 & \text{to full membership} \end{cases}$$

According to Zadeh a fuzzy subset of $X$ is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \longrightarrow [0, 1]$.

We shall mainly consider the following three spaces of nonempty subsets of $\mathbb{R}^n$ :

- $\mathrm{C}^n(\mathbb{R}^n)$ Cn consisting of all nonempty closed subsets of $\mathbb{R}^n$,
- $\mathrm{K}_n(\mathbb{R}^n)$ consisting of all nonempty compact (i.e. closed and bounded) subsets of $\mathbb{R}^n$,
- $\mathrm{P}_{kc}(\mathbb{R}^n)$ consisting of all nonempty compact convex subsets of $\mathbb{R}^n$.

Thus we have the strict inclusions

$$\mathrm{C}^n(\mathbb{R}^n) \subset \mathrm{K}_n(\mathbb{R}^n) \subset \mathrm{P}_{kc}(\mathbb{R}^n)$$

Let $x$ be a point in $\mathbb{R}^n$ and $A$ a nonempty subset of $\mathbb{R}^n$ . We define the distance $d(x, A)$ from $x$ to $A$ by

$$d(x, A) = inf\{||x - a|| : a \in A\}$$

We shall call the subset

$$S_\varepsilon(A) = \{x \in \mathbb{R}^n : d(x, A) < \varepsilon\}$$

an $\varepsilon$-neighbourhood of $A$. Its closure is the subset

$$\bar{S}_\varepsilon(A) = \{x \in \mathbb{R}^n : d(x, A) < \varepsilon\}$$

In particular, we shall denote by $\bar{S}_1^n$ the closed unit ball in $\mathbb{R}^n$, that is

$$\bar{S}_1^n = \bar{S}_1^n(\{0\})$$

Note that

$$S_\varepsilon(A) = A + \varepsilon \bar{S}_1^n$$

for any $\varepsilon > 0$ and any nonempty subset $A$ of $\mathbb{R}^n$

We define the Hausdorff separation of $B$ from $A$ by

$$d_H^*(B, A) = sup\{d(b, A) : b \in B\}$$

or, equivalently,

$$d_H^*(B, A) = inf\{\varepsilon > 0 : \ \ B \subset A + \varepsilon \bar{S}_1^n\}.$$

Thus we have $d_H^*H(B, A) > 0$ with $d_H^*H(B, A) = 0$ if and only if $B \subset \bar{A}$ In addition, the triangle inequality

$$d_H^*(A, B) \leq d_H^*(A, C) + d_H^*(C, B)$$

$A$, $B$ and $C$ subsets of $\mathbb{R}^n$.
however $$d_H^*(A, B) \neq d_H^*(B, A)$$

We define the Hausdorff distance between nonempty subsets $A$ and $B$ of $\mathbb{R}^n$ by

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

This is now symmetric in $A$ and $B$.

For $A, B \in \mathrm{P}_{kc}(\mathbb{R}^n)$ , set the Hausdorf metric as

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} \|a-b\|, d(b,A) = \inf_{a\in A} \|b-a\|.$

**Another way to see the Hausdorf distance**

Let $\sigma(A,x) = \sup_{a\in A}\{< a,x >\}$ be the support function of $A \in \mathrm{P}_{kc}(\mathbb{R}^n)$

Denote $|A| = H(A,\ \{0\})$ . In particular,

$H(A,B) = \sup\{|\sigma(A,x) - \sigma(B,x)| : x \in S^{n-1}\},$

where $S^{n-1} = \{x \in R^n : \|x\| = 1\}.$

$u : R^n \to [0, 1]$ is called a fuzzy number if it satisies

- $u$ is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;
- $u$ is fuzzy convex ;
- $u$ is upper semicontinuous ;
- $[u]^0 = \mathrm{cl}\{ \bigcup_{\alpha \in [0,1]} u^\alpha \} \in \mathrm{P}_{kc}(\mathbb{R}^n)$.

Denote $E^n = \{u : u \text{ is a fuzzy number. } \}$

We can introduce addition and scalar multiplication in $E^n$.
For $u, v \in E^n, c \in R$, set that

$$\begin{cases} [u+v]^\alpha = [u]^\alpha + [v]^\alpha \\ [cu]^\alpha = c[u]^\alpha \\ \theta : R^n \to [0,1] & \text{is defined as the null element of } E^n \\ \theta(x) = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ otherwise} \end{cases} \end{cases}$$

It is obvious that for each $u \in E^n, u + \theta = \theta + u = u$.
The metric $D$ of $E^n$ is defined as

$$D(u,\ v) = \sup_{0 \leq \alpha \leq 1} H([u]^\alpha,\ [v]^\alpha)$$

By [9 ] Puri and Ralescu, differential for fuzzy function $(E^n, D)$ is a complete metric space.

We say that a set valued mapping $F$ satisfying is continuous at to in $T \subset \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists a $\eta = \eta(\varepsilon, t_0) > 0$ such that

$$d_H(F(t), F(t_0)) < \varepsilon$$

for all $t \in T$ with $||t - t_0|| < \eta$
Alternatively, we can write in terms of the convergence of sequences, that is as

$$lim_n(F(t_n), F(t_0)) = 0$$

for all sequences $\{t_n\}$ in $T$ with $t_n \longrightarrow t_0$.

**Measurability**

Denote $T = [a,\ b]$. The mapping $F : T \to E^n$ is measurable if for each $\alpha \in [0,\ 1\ ]$ the set-valued mapping $F^\alpha : T \to \mathrm{P}(\mathrm{R})$ is measurable where $F^\alpha(t) = [F(t)]^\alpha$.

**Integrability**

We call a measurable mapping $F : T \to E^n$ integrably bounded if there exists a Lebesgue-integrable function $h(t) : T \to [0,\ +\infty)$ such that for each $x \in [F(t)]^0 \|x\| \leq h(t)$ .

For integrably bounded mapping $F$, the integral $\displaystyle\int_A F\mathrm{d}t$ of $F$ on any Lebesgue- measurable subset $A \subset T$ is defined as

$$[\int_A F\mathrm{d}t]^\alpha = \int_A F^\alpha \mathrm{d}t = \{\int_A f(t)\mathrm{d}t : f(t)\text{is an integrable selector of } [F(t)]^\alpha\},$$

where $\alpha \in [0,\ 1\ ]$.

## Differentiation

A mapping $F : T \to E^n$ is differentiable at $t_0 \in T$ if there exists $f'(t_0) \in E^n$ such that the limits

$$\begin{cases} \lim\limits_{h\to 0^+} \dfrac{F(t_0+h)-F(t_0)}{h} \\ \lim\limits_{h\to 0^+} \dfrac{F(t_0)-F(t_0-h)}{h} \end{cases}$$

exist and equal to $F'(t_0)$ . Here the limits are executed in $(E^n, D)$ .

For fixed constant $L > 0$, denote by
$U_L = \{u \in E^n : H([u]^\alpha,\ [u]^\beta) \leq L|\alpha - \beta|,$

$$\alpha,\ \beta \in [0,\ 1\ ],\ |u^0| = \sup_{x \in u^0} \|x\| \leq L\}$$

the class of Lipschitz fuzzy sets.
Let $C([0,\ 1] \times S^{n-1})$ be the space composed of all continuous functions on $[0,\ 1\ ] \times S^{n-1}$. For $g(\alpha, x) \in C([0,\ 1] \times S^{n-1})$ , define its norm as

$$\|g\|_\infty = \sup_{(\alpha,x)\in[0,1]\times \mathrm{S}^{n-1}} |g(\alpha, x)|.$$

For $u \in U_M$, define $j \circ u$ as $j \circ u(\alpha, x) = \sigma([u]^\alpha, x)$ .
By Diamond and Kloeden Metric spaces of fuzzy sets [3],
$j \circ u \in C([0,\ 1] \times S^{n-1})$ .

**Theorem (Lemma 1.)**

*(1)* $\forall u, v \in U_M, D(u,\ v) = \|j \mathrm{o} u - j \circ v\|_\infty$ *holds;*
*(2)* $j(U_M)$ *is a compact convex subset of* $C([0,1] \times S^{n-1})$ *.*

Proof. In view of [3] Diamond and Kloeden Metric spaces of fuzzy sets, we only need to show that $j(U_M)$ is closed. Let
$u_n \in U_M, j(u_n) \to g \in C([0,1] \times S^{n-1})$ .
By [14, Theorem 3 :4], there exists a subsequence $u_{n_k}$ and $u \in U_M$ such that

$$\forall(\alpha, x) \in [0,\ 1] \times S^{n-1} \sigma([u_{n_k}]^\alpha, x) \to \sigma([u]^\alpha, x)$$

.
Therefore $j(u_{n_k})(\alpha, x) \to j(u)(\alpha, x)$ and further $g = j(u) \in j(U_M)$.
**Corollary 2.** $U_M$ *is compact in* $(E^n, D)$ .

**Carathéodory's existence theorem** In mathematics, Carathéodory's existence theorem says that an ordinary differential equation has a solution under relatively mild conditions. It is a generalization of Peano's existence theorem.

A mapping $f(t,u) : [a,\ b] \times U_{M_1} \to U_M$ satisfies Caratheodory condition if

- for each $t \in [a,\ b], f(t,u)$ is continuous with respect to $u$;
- for each $u \in U_{M_1}, f(t,\ u)$ is measurable with respect to $t$.

we discuss the Caratheodory solutions for Cauchy problem of the following ordinary differential equation :

$$(1) \begin{cases} \dot{x}(t) = f(t,x) \\ x(0) = 0 \end{cases}$$

## Recall

Consider the differential equation

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} \tag{1}$$

with $f$ defined on the rectangular domain $R = \{(t, y) \mid |t - t_0| \leq a, |y - y_0| \leq b\}$. If the function $f$ satisfies the following three conditions :

- $f(t, y)$ is continuous in $y$ for each fixed $t$,
- $f$ is measurable in $t$ for each fixed $y$,
- there is a Lebesgue-integrable function $m(t)$, $|t - t_0| \leq a$, such that $|f(t, y)| \leq m(t)$ for all $(t, y) \in R$,

then the differential equation has a solution in the extended sense in a neighborhood of the initial condition.

### Definition

$x(t) : [a,\ b] \to U_{M_1}$ is called a Carathéodory solution of (1), for simplicity written as *C- solution*, if $x(t)$ is differentiable a.e. and satisfies $x(0) = \theta, \dot{x}(t) = f(t, x(t))\mathrm{a.e.}$

**Theorem (Lemma 4.)**

*Suppose that* $x(t) : [a,\ b] \to U_{M_1}$ *is continuous and* $f(t,x) : [a,\ b] \times U_{M_1} \to U_M$ *satisfies Caratheodory condition, then*

- $f(t,x(t))$ *is integrable.*
- $j \circ f(t,x(t)) : [a,\ b] \to j(U_M)$ *is Bochner-integrable.*

**Theorem**

*Let* $X$ *be a banach space and* $-\infty \prec a \prec b \prec +\infty$ *and let*
$f : [a, b] \longrightarrow X$ *a measurable function then*
$f$ *is bochner-integrable iff* $||f(.)|| \in L^1(a, b)$

Proof.

(1) As $x(t)$ is continuous, there exists a sequence of simple functions $x_n(t)$ such that $\lim\limits_{n\to\infty} x_n(t) = x(t)$ ,and then $f(t, x(t)) = \lim\limits_{n} f(t, x_n(t))$ .

Since $f(t, x)$ is measurable with respect to $t$, we have that for each $\alpha \in [0,\ 1\ ], [f(t, x_n(t))]^{\alpha} : [a,\ b] \to \mathrm{P}(\mathbb{R}^n)$ is measurable.

By [11, Property 3 :8], for each $x \in S^{n-1}$ the support function $\sigma([f(t, x_n(t))]^{\alpha}, x)$ is measurable and further $\sigma([f(t, x(t))]^{\alpha}, x)$ is measurable.

Therefore $[f(t, x(t))]^{\alpha}$ is measurable, that is to say, $f(t, x(t))$ is measurable. In view of the fact that $f(t, x(t))$ is integrably bounded, we conclude that $f(t, x(t))$ is integrable.

(2) By [5, Theorem 3 :1], $f(t, x(t))$ is measurable with respect to the metric $D$. From Corollary 2, for $\forall \epsilon > 0$, there exists finite $\epsilon$-net $\{u_1,\ u_2,\ \cdots,\ u_n\}$ of $U_M$.

Denote $A_j = \{t : D(f(t, x(t)),\ u_j) < \epsilon\}$. Then $A_j$ is measurable set and $\bigcup_{i=1}^{n} A_j = [a,\ b]$.

Denote $B_1 = A_1, B_j = A_j \setminus \bigcup_{j=1}^{i-1} A_j(\mathrm{i} \geq 2)$ .

Then $\{B_j\}$ are disjoint and measurable, and $\bigcup_{i=1}^{n} B_j = [a,\ b]$.

Set

$$f^{\epsilon}(t) = \sum_{i=1}^{n} u_j \psi_{B_j}(t) \ ,$$

where $\psi_{B_j}(t)$ denotes the characteristic function of $B_j$. Since $D(f(t,x(t)),\ f^{\epsilon}(t)) < \epsilon$,
we have $\|j \circ f(t,x(t)) - j \circ f^{\epsilon}(t)\|_{\infty} < \epsilon$, which implies that $j \circ f(t,x(t))$ can be approximated by simple functions. So $j \circ f(t,x(t))$ is strongly measurable, and furthermore Bochner-integrable.

## Theorem (Lemma 5.)

*If* $F(t) : T \to U_M$ *is measurable, then* $G(t) = \int_a^t F(s)\mathrm{d}s$ *satisfies*

- $G(t) : T \to U_{M_1}$ *where* $M_1 = M(b-a)$ *;*
- $G'(t) = F(t)a.e.$

Proof.

- $H([G(t)]^\alpha,[G(t)]^\beta)=H(\int_a^t[F(s)]^\alpha ds,\int_a^t[F(s)]^\beta \mathrm{d}s)$
  $=\int_a^t H([F(s)]^\alpha,[F(s)]^\beta)\mathrm{d}s$
  $\leq M(t-a)$
  $\leq M(b-a)=M_1,$
  $|[G(t)]^0|=|\int_a^t[F(s)]^0\mathrm{d}s|$
  $$\leq M(t-a)\leq M(b-a)=M_1.$$
  So $G(t):T\to U_{M_1}.$
- Similar to the proof of Lemma 4, we obtain that $j\circ F(s)$ is Bochner-integrable and for each $h>0$ we have
  $D(\frac{G(t+h)-G(t)}{h},F(t))$
  $=D(\frac{1}{h}\int_t^{t+h}F(s)\mathrm{d}s,F(t))$

$$= D(\frac{1}{h}\int_t^{t+h} F(s)\mathrm{d}s, \frac{1}{h}\int_t^{t+h} F(t)\mathrm{d}s)$$
$$\leq \frac{1}{h}\int_t^{t+h} D(F(s), F(t))\mathrm{d}s$$
$$= \frac{1}{h}\int_t^{t+h} \|j\circ F(s) - j\circ F(t)\|_\infty \mathrm{d}s.$$

As $\lim_{h\to 0} 1/h \int_t^{t+h} \|j\circ F(s) - j\circ F(t)\|_\infty = 0$ a.e.,
we have $G'(t) = F(t)\mathrm{a.e.}\Box$

**Theorem (Lemma 6.)**

$j(\mathcal{K}) = \{j \circ u(t) : u \ \in \mathcal{K}\}$ *is compact convex in* $C([a,\ b], C([0,\ 1] \times S^{n-1}))$ *where*
$\mathcal{K} = \{u(t) = \int_a^t v(s)\mathrm{d}s : v : [a,\ b] \to U_M \text{is integrable}\}$.

Proof. If $u(t) \in \mathcal{K}$, then $u(t) = \int_a^t v(s)\mathrm{d}s$. Similar to the proof of Lemma 4, we get that $j \circ v(s) : [a,\ b] \to C([0,\ 1] \times S^{n-1})$ is strongly measurable and Bochner-integrable. By [5, Theorem 2 :2], we have

$$j \circ u(t) = \int_a^t j \circ v(s)\mathrm{d}s$$

and further $j \circ u(t) \in C([a, b], C([0,\ 1] \times S^{n-1}))$

- From $D(u(t), \theta) = D(\int_a^t v(s)\mathrm{d}s, \theta) \leq \int_a^t D(v(s), \theta)\mathrm{d}s \leq M(b-a) = M_1$
  we know that
  $\|j \circ u(t)\|_\infty = D(u(t),\ \theta) \leq M_1$ and
  $\sup_{t\in[a,b]} \|j \circ u(t)\|_\infty \leq M_1$,
  that is to say, $j(K)$ is uniformly bounded.
- For $t_1 < t_2 \in [a,\ b]$,

$$D(u(t_2), u(t_1)) = D(\int_a^{t_2} v(s)\mathrm{d}s, \int_a^{t_1} v(s)\mathrm{d}s) =$$

$$D(\int_{t_1}^{t_2} v(s)\mathrm{d}s, \theta)$$

$$\leq \int_{t_1}^{t_2} D(v(s),\ \theta)\mathrm{d}s$$

$$\leq M(t_2 - t_1)\ .$$

Then $\|j \circ u(t_2) - j \circ u(t_1)\|_\infty \leq M(t_2 - t_1)$ , or $j(K)$ is equicontinuous.

- By Lemma 5, for each $t \in [a,\ b], \{u(t)\}_{u \in \mathrm{K}} \subset U_{M_1}(M_1 = M(b - a$ so $\{j \circ u(t)\}_{u \in \mathrm{K}} \subset j(U_{M_1})$ . By Lemma 1, $\{j \circ u(t)\}_{u \in \mathrm{K}}$ is relatively compact in $C([0, 1] \times S^{n-1})$ .

the conditions of Ascoli-Arzela theorem are satisfied, therefore $j(K)$ is relatively compact in $C([a,\ b], C([0,\ 1] \times S^{n-1}))$

It is obvious that $j(K)$ is convex.

Next we need to only show that $j(K)$ is closed.

Suppose that $u_n(t) \in \mathcal{K}$ and
$j \circ u_n(t) \to f(t) \in C([a,b], C([\mathrm{O},\ 1] \times S^{n-1}))$ Denote
$u_n(t) = \displaystyle\int_a^t v_n(s)\mathrm{d}s$ where $v_n(s) : [a,\ b] \to U_M$ is integrable.
By [14, Theorem 4 :2],
there exists a subsequence $\{v_{n_k}(s)\}$ of $\{v_n(s)\}$ and
$v(s) : [a,\ b] \to U_M$ such that
$\sigma([\displaystyle\int_a^t v_{n_k}(s)\mathrm{d}s]^\alpha, x) \to \sigma([\displaystyle\int_a^t v(s)\mathrm{d}s]^\alpha, x)$ ,
where $(\alpha, x) \in [0,1] \times S^{n-1}$.
This implies that $jou_{n_k}(t)(\alpha, x) \to jou(t)(\alpha, x)$ .
Then $f(t)(\alpha, x) = j \circ u(t)(\alpha, x)$ or $f(t) = j \circ u(t)$ ,
$u(t) = \displaystyle\int_a^t v(s)\mathrm{d}s$, furthermore $f(t) \in j(\mathcal{K})$ . $\square$

**Theorem**

*If* $f(t,x) : [a,\ b] \times U_{M_1} \to U_M$ *satisfies Caratheodory condition, then (1) has at least one* $C$*-solution.*

Proof. Let $\mathcal{K}$ be the same as in Lemma 6. For $u \in \mathcal{K}$, define $h(u)(t) = \int_a^t f(t,\ u(s))\mathrm{d}s$. From Lemmas 4 and 5, we know that $h(u) \in \mathcal{K}$ and so $H : \mathcal{K} \to \mathcal{K}$.
On account of that, the mapping $T = j \circ h \circ j^{-1} : j(\mathcal{K}) \to j(K)$ is $T(j(u))$
$= \int_a^t j \circ f(s, u(s))\mathrm{d}s.$
To prove that $T$ is continuous, we need to only show that $h$ is continuous.
Suppose that $u_n \to u$, then $u_n(s) \to u(s)$ for each $s \in [a,\ b]$.
As $f$ satisfies Caratheodory condition.
$f(s,\ u_n(s)) \to f(s,\ u(s))$ a.e., that is to say,
$D(f(s, u_n(s)), f(s, u(s))) \to 0$ a.e.
we have

$$\sup_{t\in[a,b]} D(h(u_n)(t), h(u)(t)) \leq \int_a^b D(f(s, u_n(s)), f(s, u(s)))\ \mathrm{ds} :$$

and since

$D(f(s,\ u_n(s)),\ f(s,\ u(s))) \leq 2M,$

by Lebesgue-dominated convergence theorem,

$\int_a^b D(f(s,\ u_n\ f(s,\ u(s))\mathrm{d}s \to 0$ and moreover

$\sup_{t\in[a,b]} D(h(u_n)(t),\ h(u)(t)) \to 0.$

By Lemma 6, since the operator $T$ satisfies the conditions of Schauder fixed point theorem, there exists $u_* \in \mathcal{K}$ such that

$T(j(u_*)) = j(u_*)$ or

$$j(u_*)(t) = \int_a^t j \circ f(s,\ u_*(s))\ \mathrm{ds}:$$

Therefore $u_*(t) = \int_a^t f(s,\ u_*(s))\mathrm{d}s$, finaly we get

$u_*'(t) = f(t,\ u_*(t))\mathrm{a.e.}$ which means that $u_*(t)$ is a solution of (1).

□

## References :

[1] D. Butnariu, Measurability concepts for fuzzy mappings, Fuzzy Sets and Systems 31 (1989) 7782.

[2] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Springer, Berlin, 1977.

[3] P. Diamond, P.E. Kloeden, Metric Spaces of Fuzzy Sets : Theory and Applications, World Scientific, Singapore, 1994.

[4] M. Hukuhara, Integration des applications measurables dont la valeur est compact convexe, Funkcialaj Ekvacioj 10 (1967) 205223.

[5] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301317.

[6] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems 35 (1990) 309316.

[7] A. Kandel, W.J. Byatt, Fuzzy differential equations, Proceedings of the International Conference on Cybernetics and Society, Tokyo, November 1978, pp. 12131216.

[8] P.E. Kloeden, Remarks and Peano-like theorems for fuzzy

differential equations, Fuzzy Sets and Systems 44 (1991) 161163.

[9] M.L. Puri, D.A. Ralescu, Differentials for fuzzy function, J. Math. Anal. Appl. 91 (1983) 552558.

[10] M.L. Puri, D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409422.

[11] N. Papageorgiou, On the theory of Banach space valued multifunctions, J. Multivariate Anal. 17 (1985) 185206.

[12] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319330.

[13] Xue Xiaoping, Ha Minghu, Ma Ming, Random fuzzy number integrals in Banach spaces, Fuzzy Sets and Systems 66 (1994) 97111.

[14] Xue Xiaoping, Wang Xiaomin, Wu Lizhong, On the convergence and representation of random fuzzy number integrals, Fuzzy Sets and Systems 103 (1999) 115125.

Thank you for your attention